

Cohomology of GKM-sheaves

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Abstract

Let T be a compact torus and X be a finite T -CW complex (e.g. a compact T -manifold). In [2], the second author introduced a functor which assigns to X a so called GKM-sheaf \mathcal{F}_X whose ring of global sections $H^0(\mathcal{F}_X)$ is isomorphic to the equivariant cohomology $H_T^*(X)$ whenever X is equivariantly formal (meaning that $H_T^*(X)$ is a free module over $H^*(BT)$). In the current paper we prove more generally that $H^0(\mathcal{F}_X) \cong H_T^*(X)$ if and only if $H_T^*(X)$ is reflexive, and find a geometric interpretation of the higher cohomology $H^n(\mathcal{F}_X)$ for $n \geq 1$.

1 Introduction

Let $T = (\mathbb{S}^1)^r$ be a compact torus Lie group and let X be a finite T -CW complex (such as a compact, smooth T -manifold). GKM theory provides techniques for computing the equivariant cohomology ring $H_T^*(X) := H^*(ET \times_T X; \mathbb{C})$. For a large class of T -manifolds, now called GKM-manifolds, Goresky, Kottwitz and MacPherson [5] showed that the cohomology ring

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$H_T^*(X)$ can be encoded combinatorially in a finite graph (the GKM-graph or moment graph) with edges labelled by non-trivial weights $\alpha \in \Lambda := \text{Hom}(T, \mathbb{S}^1)$. GKM-theory has since developed in several directions: combinatorially by Guillemin and Zara [7–9], to a broader range of spaces by Guillemin and Holm [6], and to equivariant intersection cohomology by Braden and MacPherson [3] who introduced the notion of Γ -sheaves on a GKM-graph.

In [2] the second author introduced GKM-sheaves which provide a unified framework for the above constructions. Given a finite T -CW complex X , we associate a sheaf \mathcal{F}_X whose ring of global sections $H^0(\mathcal{F}_X)$ is isomorphic to $H_T^*(X)$ whenever $H_T^*(X)$ is a free module over the cohomology of a point. In the current paper we improve this result by proving that $H^0(\mathcal{F}_X) \cong H_T^*(X)$ if and only if $H_T^*(X)$ is a reflexive module (equivalently a 2-syzygy). Furthermore we show that $H^n(\mathcal{F}_X) = 0$ for $n \geq 2$ and that

Theorem 1.1. *If X is a finite T -CW complex and $H_T^*(X)$ is reflexive, then there is a natural exact sequence*

$$0 \rightarrow H^0(\mathcal{F}_X) \rightarrow H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \rightarrow H^1(\mathcal{F}_X) \rightarrow 0$$

where $X_0 = X^T$ is the fixed point set, X_1 is the union of all orbits of dimension one or less, and δ is the coboundary map in the long exact sequence of the pair (X_1, X_0) .

Outline: In §2, §3, and §4 we review background material on equivariant cohomology, GKM sheaves, and sheaf cohomology respectively. In §5 we study the cohomology of GKM-sheaves and prove that $H^n(\mathcal{F}) = 0$ for $n \geq 0$ and produce chain complexes to calculate $H^1(\mathcal{F})$. In §6 we study the cohomology of the GKM-sheaf \mathcal{F}_X associated to T -space and interpret it geometrically.

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2 Equivariant Cohomology

Let $T \cong (\mathbb{S}^1)^r$ be a compact torus Lie group. The *universal T -bundle*, $ET \rightarrow BT$ is a principal bundle for which ET is contractible and whose base BT is homotopy equivalent to the r -fold product space $\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$. Given a T -space X , the *Borel homotopy quotient* $X_{hT} := ET \times_T X$ is the total space of the associated fibre bundle

$$X \xrightarrow{i} ET \times_T X \xrightarrow{\pi} BT. \tag{1}$$

The *equivariant cohomology* of X is the singular cohomology of the homotopy quotient

$$H_T^*(X) := H^*(X_{hT}; \mathbb{C}).$$

We use complex coefficients throughout. More generally, if $Y \subseteq X$ is a T -invariant subspace then

$$H_T^*(X, Y) = H^*(X_{hT}, Y_{hT}).$$

Given a T -space X , the constant map to a point $X \rightarrow pt$ is equivariant. The induced morphism $H^*(pt) \rightarrow H_T^*(X)$ makes $H_T^*(X)$ an algebra over $H_T^*(pt) = H^*(BT)$. By the Kunneth formula

$$H^*(BT) = H^*(\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty) \cong \mathbb{C}[x_1, \dots, x_r]$$

where each class x_i has degree two. More invariantly, there is a natural isomorphism between $H^*(BT)$ and the ring of complex valued polynomial functions on the Lie algebra \mathfrak{t}

$$H^*(BT) \cong \mathbb{C}[\mathfrak{t}].$$

The *weight lattice*

$$\Lambda := \{\alpha : T \rightarrow \mathbb{S}^1\} \tag{2}$$

is the set of Lie group homomorphisms from T to $\mathbb{S}^1 = U(1)$. It forms a group under multiplication and there is a natural injection

$$\Lambda \hookrightarrow H^2(BT)$$

which sends $\alpha \in \Lambda$ to the tangent map $d\alpha : \mathfrak{t} \rightarrow \mathfrak{u}(1) = i\mathbb{R}$ regarded as a homogeneous linear polynomial in $\mathbb{C}[\mathfrak{t}]$.

The Borel Localization Theorem is central to GKM theory. We require only the following basic version (see [4]).

Theorem 2.1 (Localization Theorem). *Let X be a finite T -CW complex with fixed point set $i : X^T \hookrightarrow X$. Then the kernel and cokernel of $i^* : H_T^*(X) \rightarrow H_T^*(X^T)$ are both torsion $H^*(BT)$ -modules. In particular if $H_T^*(X)$ is torsion free then i^* is injective.*

2.1 Atiyah-Bredon Sequence

Let $R = H^*(BT) \cong \mathbb{C}[x_1, \dots, x_r]$. A finitely generated R -module M is said to be a j -th syzygy if there exists an exact sequence

$$0 \rightarrow M \rightarrow F^1 \rightarrow F^2 \rightarrow \dots \rightarrow F^j$$

where the $\{F^i\}_{i \in \{1, \dots, j\}}$ are finitely generated free R -modules. According to ([1] Prop. 2.3):

- M is an r -syzygy if and only if M is free,
- M is a 1-syzygy if and only if M is torsion free,
- M is a 2-syzygy if and only if M is reflexive, meaning the natural map

$$M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$$

is an isomorphism.

Given a T -space X , define X_i to be the union of all orbits of dimension less than or equal to i ,

$$X_i := \{x \in X \mid \dim(T \cdot x) \leq i\}.$$

We call X_i the i -**skeleton** of X . In particular, $X_{-1} = \emptyset$, $X_0 = X^T$, and $X_r = X$.

The following is due to Allay-Franz-Puppe ([1] Theorem 5.7).

Theorem 2.2. *Let $j \geq 0$ and let T be a torus of rank r , and X be a finite T -CW complex. Consider the sequence*

$$0 \rightarrow H_T^*(X) \rightarrow H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_r} H_T^{*+r}(X_r, X_{r-1}) \rightarrow 0, \quad (3)$$

where δ_i is the boundary map of the triple (X_{i-1}, X_i, X_{i+1}) . Then (3) is exact for all the positions $i \leq j - 2$ if and only if $H_T^*(X)$ is j -th syzygy. In particular, the sequence

$$0 \rightarrow H_T^*(X) \rightarrow H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \quad (4)$$

is exact if and only if $H_T^*(X)$ is a 2-syzygy if and only if $H_T^*(X)$ is reflexive.

The sequences (3) and (4) are known as the Atiyah-Bredon sequence and the Chang-Skjelbred sequence respectively. Observe that if (4) is exact, then

$$H_T^*(X) \cong \ker(\delta).$$

GKM theory is concerned with calculating $\ker(\delta)$.

3 GKM-Sheaves

Recall the weight lattice $\Lambda := \text{Hom}(T, \mathbb{S}^1)$ from (2). Declare two weights α, β to be projectively equivalent if $\alpha^n = \beta^m$ for some $m, n \in \mathbb{Z}$. The set of *projective weights* $\mathbb{P}(\Lambda)$ is the set of non-zero weights in Λ modulo projective equivalence. The elements of $\mathbb{P}(\Lambda)$ are in one to one correspondence with the codimension one subtori of T by the rule

$$\alpha \in \mathbb{P}(\Lambda) \leftrightarrow \ker_0(\tilde{\alpha}) \leq T$$

where $\tilde{\alpha} \in \Lambda$ is a representative of α , and $\ker_0(\tilde{\alpha})$ is the identity component of the kernel of $\tilde{\alpha} : T \rightarrow S^1$. We denote $\ker(\alpha) = \ker_0(\tilde{\alpha})$.

Definition 1. A **GKM-hypergraph** Γ consists of:

1. A finite set of vertices $\mathcal{V} = \mathcal{V}_\Gamma$.
2. An equivalence relation \sim_α on \mathcal{V} for each $\alpha \in \mathbb{P}(\Lambda)$.

Given a GKM-hypergraph Γ , the set of hyperedges is defined to be

$$\mathcal{E} = \mathcal{E}_\Gamma := \{(S, \alpha) \in \wp(\mathcal{V}) \times \mathbb{P}(\Lambda) \mid S \text{ is an equivalence class for } \sim_\alpha\}$$

where $\wp(V)$ is the power set of \mathcal{V} . We have projection maps

- $a : \mathcal{E} \rightarrow \mathbb{P}(\Lambda)$ the *axial function*, and
- $I : \mathcal{E} \rightarrow \wp(\mathcal{V})$ the *incidence function*.

In particular, each hyperedge $e \in \mathcal{E}$ has associated to it a projective weight $a(e)$ and a non-empty subset $I(e) \subseteq \mathcal{V}$. We say a vertex $v \in \mathcal{V}$ is *incident* to $e \in \mathcal{E}$ if $v \in I(e)$. Given $\alpha \in \mathbb{P}(\Lambda)$ denote by $\mathcal{E}_\alpha := \{e \in \mathcal{E} \mid a(e) = \alpha\}$.

Let $\text{Top}(\Gamma)$ to be the topological space with underlying set $\mathcal{V} \cup \mathcal{E}$ generated by basic open sets $U_v = \{v\}$ for $v \in \mathcal{V}$, and $U_e = \{e\} \cup I(e)$ for $e \in \mathcal{E}$. Observe that for each $x \in \text{Top}(\Gamma)$, the set U_x is smallest open set containing x .

Definition 2. Let $R := H^*(BT) \cong \mathbb{C}[\mathfrak{t}]$. A GKM-sheaf \mathcal{F} is a sheaf of finitely generated, \mathbb{Z} -graded R -modules over $\text{Top}(\Gamma)$, satisfying the following conditions.

1. \mathcal{F} is locally free (i.e, for every basic open set U_x , the stalk $\mathcal{F}(U_x) = \mathcal{F}_x$ is a free R -module).
2. For every hyperedge $e \in \mathcal{E}_\Gamma$, the restriction map $res_e : \mathcal{F}(U_e) \rightarrow \mathcal{F}(I(e))$ is an isomorphism upon inverting $a(e)$:

$$\mathcal{F}(U_e) \otimes_R R[a(e)^{-1}] \cong \mathcal{F}(I(e)) \otimes_R R[a(e)^{-1}].$$

3. The restriction map $res_e : \mathcal{F}(U_e) \rightarrow \mathcal{F}(I(e))$ is an isomorphism for all but a finite number of $e \in \mathcal{E}_\Gamma$.

The main motivating example is the GKM-hypergraph Γ_X and GKM-sheaf \mathcal{F}_X associated to a finite T -CW complex X . The vertex set of Γ_X is the set $\mathcal{V}_X := \pi_0(X^T)$ of path components of the T -fixed point set X^T . Define $v_1 \sim_\alpha v_2$ if and only v_1 and v_2 lie in the same path component of the fixed point set $X^{\ker(\alpha)}$. The hyperedges $e \in \mathcal{E}_\alpha := \{e \in \mathcal{E} \mid a(e) = \alpha\}$ therefore correspond to path components of $X^{\ker(\alpha)}$ that intersect X^T non-trivially.

Define the GKM-sheaf \mathcal{F}_X over Γ_X , as follows. The stalk at a vertex $v \in \pi_0(X^T)$ is

$$\mathcal{F}_X(U_v) := H_T^*(v),$$

and at a hyperedge $e \in \pi_0(X^{\ker(\alpha)})$ is

$$\mathcal{F}_X(U_e) = \mathcal{F}_X(e \cup I(e)) = H_T^*(e)/t,$$

where t is the torsion submodule of $H_T^*(e)$. The sheaf restriction maps $res_e : \mathcal{F}_X(U_e) \rightarrow \mathcal{F}_X(I(e))$ are identified with the natural map $H_T^*(e)/t \rightarrow H_T^*(e^T)$ which is well defined because $e^T \subset e$ and $H_T^*(e^T)$ is torsion free. This data completely determines \mathcal{F}_X .

The following result (Proposition 2.7 in [2]), relates the degree zero sheaf cohomology of \mathcal{F}_X with the equivariant cohomology of X .

Proposition 3.1. *Let X be a finite T -CW complex. The space of global sections $H^0(\mathcal{F}_X)$ fits into an exact sequence of graded R -modules*

$$0 \rightarrow H^0(\mathcal{F}_X) \xrightarrow{r} H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0). \quad (5)$$

We obtain a generalization of the main result of [2], which was originally proven only when $H_T^*(X)$ is a free module.

Corollary 3.2. *Let X be a finite T -CW complex. If $H_T^*(X)$ is reflexive, then*

$$H_T^*(X) \cong H^0(\mathcal{F}_X).$$

Proof. Combine Proposition 3.1 with the Chang-Skjelbred sequence (4) which holds if $H_T^*(X)$ is reflexive. \square

For later use, we state the following lemmas from [2].

Lemma 3.3. *If X is a finite T -CW complex and $H \subset T$ is a codimension one subtorus, then $H_T^*(X^H)$ is the direct sum of a free and a torsion R -module. If $H_T^*(X)$ is torsion free, then $H_T^*(X^H)$ is free.*

Proof. This is Lemma 2.6 in [2]. \square

Lemma 3.4. *Let X'_1 be the union of those components of X_1 which do not intersect with X_0 . Then $H_T^*(X_1, X_0)$ decomposes into*

$$H_T^*(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H_T^*(e, e^T) \oplus H_T^*(X'_1). \quad (6)$$

Proof. This is Proposition 2.7 in [2]. \square

4 Sheaf Cohomology using the Godement Resolution

We summarize material from Iversen [10]. Given a sheaf \mathcal{F} over a topological space Y , define the sheaf $C^0\mathcal{F}$ which sends open sets $U \subseteq Y$ to be the product of stalks

$$C^0\mathcal{F}(U) = \prod_{y \in U} \mathcal{F}_y \quad (7)$$

and whose restriction morphisms are given by projection. There is a natural monomorphism of sheaves,

$$\mathcal{F} \longrightarrow C^0\mathcal{F} \quad (8)$$

which sends $s \in \mathcal{F}(U)$ to the product of germs $(s_y)_{y \in U} \in C^0(\mathcal{F})(U)$. Construct sheaves \mathcal{F}^n for all $n \geq 0$ inductively by setting $\mathcal{F}^0 := \mathcal{F}$ and setting \mathcal{F}^n equal to the cokernel sheaf of the natural monomorphism $\mathcal{F}^{n-1} \rightarrow C^0\mathcal{F}^{n-1}$ for all $n \geq 1$. Denote $C^n\mathcal{F} = C^0\mathcal{F}^n$. By construction we get short exact sequences of sheaves

$$0 \rightarrow \mathcal{F}^n \xrightarrow{f_n} C^n\mathcal{F} \xrightarrow{g_n} \mathcal{F}^{n+1} \rightarrow 0$$

for all $n \geq 0$. Let $d_n := f_{n+1} \circ g_n$ be the composition,

$$C^n\mathcal{F} \xrightarrow{g_n} \mathcal{F}^{n+1} \xrightarrow{f_{n+1}} C^{n+1}\mathcal{F} \quad (9)$$

Theorem 4.1. *The sequence of sheaves*

$$0 \rightarrow \mathcal{F} \rightarrow C^0\mathcal{F} \xrightarrow{d_0} C^1\mathcal{F} \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n\mathcal{F} \xrightarrow{d_n} \dots \quad (10)$$

is exact. It is called the Godement resolution of \mathcal{F} .

Proof. See [10]. □

Given an open set $U \subseteq Y$, define the chain complex

$$0 \xrightarrow{d_{-1}} C^0\mathcal{F}(U) \xrightarrow{d_0} C^1\mathcal{F}(U) \xrightarrow{d_1} \dots \xrightarrow{d_n} C^n\mathcal{F}(U) \rightarrow \dots$$

which satisfies $d_n \circ d_{n-1} = 0$ for all $n \geq 0$. Define the degree n cohomology of \mathcal{F} on U by

$$H^n(U; \mathcal{F}) := \frac{\ker(d_n)}{\text{im}(d_{n-1})}.$$

We use shorthand $H^n(\mathcal{F}) := H^n(Y, \mathcal{F})$. The sequence

$$0 \rightarrow \mathcal{F}(U) \rightarrow C^0\mathcal{F}(U) \xrightarrow{d_0} C^1\mathcal{F}(U)$$

is exact, which implies $H^0(U, \mathcal{F}) = \mathcal{F}(U)$.

Definition 3. For a sheaf \mathcal{F} on Y and a closed subset $A \subseteq Y$ we define

$$\Gamma_A(\mathcal{F}) = \{s \in \mathcal{F}(Y) \mid \text{supp}(s) \subseteq A\}$$

where $\text{supp}(s) = \{y \in Y \mid s_y \neq 0\}$. If $A = \emptyset$ we write $\Gamma_\emptyset(\mathcal{F}) = \Gamma(\mathcal{F}) = \mathcal{F}(Y)$.

Given the chain complex

$$0 \xrightarrow{d_{-1}} \Gamma_A(C^0\mathcal{F}) \xrightarrow{d_0} \Gamma_A(C^1\mathcal{F}) \xrightarrow{d_1} \dots \xrightarrow{d_n} \Gamma_A(C^n\mathcal{F}) \rightarrow \dots$$

define the local cohomology $H_A^n(Y, \mathcal{F}) := \frac{\ker(d_n)}{\text{im}(d_{n-1})}$.

Proposition 4.2. *Let A be closed in Y . A sheaf \mathcal{F} on Y gives rise to a long exact sequence*

$$0 \rightarrow H_A^0(Y, \mathcal{F}) \rightarrow H^0(Y, \mathcal{F}) \rightarrow H^0(Y - A, \mathcal{F}) \xrightarrow{\delta} H_A^1(Y, \mathcal{F}) \rightarrow H^1(Y, \mathcal{F}) \rightarrow \dots$$

Proof. See Proposition 9.2 in [10]. □

5 Cohomology of GKM-Sheaves

5.1 The Godement Chain Complex for GKM-Sheaves

Proposition 5.1. *If Γ is a GKM-hypergraph and \mathcal{F} is a sheaf on $\text{Top}(\Gamma)$, then $C^n\mathcal{F} = 0$, for all $n \geq 2$.*

Proof. The basic open sets of $\text{Top}(\Gamma)$ are

- (i) $U_v := \{v\}$ for vertices $v \in \mathcal{V}$.
- (ii) $U_e := \{e\} \cup I(e)$ for hyperedges $e \in \mathcal{E}$.

Given a sheaf \mathcal{F} on $Top(\Gamma)$, for each vertex v we have

$$(C^0\mathcal{F})_v := \prod_{x \in U_v} \mathcal{F}_x = \mathcal{F}_v.$$

The induced morphism $\mathcal{F}_v \rightarrow (C^0\mathcal{F})_v = \mathcal{F}_v$, is an isomorphism, so the cokernel $\mathcal{F}_v^1 = C^1\mathcal{F}_v$ is zero. Similarly, for all $n \geq 1$,

$$\mathcal{F}_v^n = C^n\mathcal{F}_v = 0. \quad (11)$$

For each hyperedge e , \mathcal{F}_e^2 is the cokernel of the product of restriction maps

$$\mathcal{F}_e^1 \rightarrow \prod_{x \in U_e} \mathcal{F}_x \cong \mathcal{F}_e^1 \times \mathcal{F}_{v_1}^1 \times \dots \times \mathcal{F}_{v_k}^1.$$

This is an isomorphism because $\mathcal{F}_{v_i}^1 = 0$, so the cokernel $\mathcal{F}_e^2 \cong 0$. We conclude $\mathcal{F}^2 = 0$ since all of its stalks vanish and consequently $C^n\mathcal{F} = 0$ for all $n \geq 2$. \square

Corollary 5.2. *If \mathcal{F} is a sheaf over $Top(\Gamma)$, then $H^n(\mathcal{F}) = 0$ for $n \geq 2$ and we have a natural exact sequence*

$$0 \rightarrow H^0(\mathcal{F}) \rightarrow \Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) \rightarrow H^1(\mathcal{F}) \rightarrow 0.$$

Proof. By Proposition 5.1, $C^n(\mathcal{F}) = \mathcal{F}^n = 0$ for all $n \geq 2$, the chain complex for $H^n(\mathcal{F})$ is

$$0 \rightarrow \Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) \rightarrow 0 \rightarrow 0 \rightarrow \dots. \quad (12)$$

\square

We want a more concrete description of (12).

Lemma 5.3. *The stalks of \mathcal{F}^1 are as follows: $\mathcal{F}_v^1 = 0$ for all vertices v , and*

$$\mathcal{F}_e^1 \cong \prod_{i=1}^k \mathcal{F}_{v_i} = \mathcal{F}(I(e))$$

for all hyperedges e , where $I(e) = \{v_1, v_2, \dots, v_k\}$ is the set of vertices incident to e .

Proof. That $\mathcal{F}_v^1 = 0$ was proven in (11). Given a hyperedge e , recall $U_e := \{e\} \cup I(e)$. The stalk \mathcal{F}_e^1 is the cokernel of the map

$$\mathcal{F}_e \xrightarrow{\varepsilon} \prod_{x \in U_e} \mathcal{F}_x = \mathcal{F}_e \times \mathcal{F}_{v_1} \times \dots \times \mathcal{F}_{v_k}$$

where $\varepsilon(s_e) = (s_e, res_{(e,v_1)}(s_e), \dots, res_{(e,v_k)}(s_e))$, where $res_{(e,v)} : \mathcal{F}_e \rightarrow \mathcal{F}_v$ is the sheaf restriction map associated to $U_v \subseteq U_e$. Because the first coordinate function of ε is the identity map on \mathcal{F}_e we obtain the isomorphism

$$\mathcal{F}_e^1 := coker(\varepsilon) \cong \prod_{i=1}^k \mathcal{F}_{v_i}$$

simply by projecting onto the remaining factors. \square

Proposition 5.4. *Let \mathcal{F} be a sheaf over $\text{Top}(\Gamma)$. Then there is a commutative diagram,*

$$\begin{array}{ccc} \Gamma(C^0\mathcal{F}) & \xrightarrow{\delta} & \Gamma(C^1\mathcal{F}) \\ \downarrow \cong & & \downarrow \cong \\ \prod_{x \in \mathcal{V} \cup \mathcal{E}} \mathcal{F}_x & \xrightarrow{\tilde{\delta}} & \prod_{e \in \mathcal{E}} \prod_{v \in I(e)} \mathcal{F}_v \end{array}$$

where the vertical maps isomorphisms, δ is as above and $\tilde{\delta}$ sends $s = (s_x)_{x \in \mathcal{V} \cup \mathcal{E}}$ to $\tilde{\delta}(s)$ with factors

$$\tilde{\delta}(s)_{(e,v)} = \text{res}_{(e,v)}(s_e) - s_v. \quad (13)$$

In particular, $H^0(\mathcal{F}) \cong \ker(\tilde{\delta})$ and $H^1(\mathcal{F}) \cong \text{coker}(\tilde{\delta})$.

Proof. The isomorphism $\Gamma(C^0\mathcal{F}) \cong \prod_{x \in \mathcal{V} \cup \mathcal{E}} \mathcal{F}_x$ is the defining identity (7). By Lemma 5.3 we have

$$\Gamma(C^1\mathcal{F}) := \prod_{x \in \mathcal{V} \cup \mathcal{E}} \mathcal{F}_x^1 = \prod_{e \in \mathcal{E}} \mathcal{F}_e^1 \cong \prod_{e \in \mathcal{E}} \prod_{i=1}^k \mathcal{F}_{v_i}.$$

The formula for $\tilde{\delta}$ is obtained by chasing through definition 9. \square

Proposition 5.5. *Let \mathcal{F} be a GKM-sheaf and let $\mathcal{E}^{nd} \subseteq \mathcal{E}$ be the finite set of hyperedges for which res_e is not an isomorphism. Then there is an exact sequence*

$$0 \rightarrow H^0(\mathcal{F}) \rightarrow \bigoplus_{x \in \mathcal{E}^{nd}} \mathcal{F}_x \xrightarrow{\beta} \bigoplus_{e \in \mathcal{E}^{nd}} \bigoplus_{v \in I(e)} \mathcal{F}_v \rightarrow H^1(\mathcal{F}) \rightarrow 0 \quad (14)$$

where β is the morphism of finitely generated free R -modules defined by

$$\beta(s)_{(e,v)} = \text{res}_{(e,v)}(s_e) - s_v.$$

Proof. Let $\mathcal{E}^d := \mathcal{E} \setminus \mathcal{E}^{nd}$. We have a commuting diagram of R -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_{x \in \mathcal{E}^d} \mathcal{F}_x & \xrightarrow{\psi} & \prod_{x \in \mathcal{V} \cup \mathcal{E}} \mathcal{F}_x & \xrightarrow{\phi} & \bigoplus_{x \in \mathcal{E}^{nd}} \mathcal{F}_x \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow \tilde{\delta} & & \downarrow \beta \\ 0 & \longrightarrow & \prod_{e \in \mathcal{E}^d} \prod_{v \in I(e)} \mathcal{F}_v & \xrightarrow{\psi'} & \prod_{e \in \mathcal{E}} \prod_{v \in I(e)} \mathcal{F}_v & \xrightarrow{\phi'} & \bigoplus_{e \in \mathcal{E}^{nd}} \bigoplus_{v \in I(e)} \mathcal{F}_v \longrightarrow 0. \end{array}$$

Where ϕ, ϕ' are projections, ψ, ψ' are inclusions, and γ is defined by commutativity. By the Snake Lemma there is an exact sequence

$$0 \rightarrow \ker \gamma \xrightarrow{\bar{\psi}} \ker(\tilde{\delta}) \xrightarrow{\bar{\phi}} \ker(\beta) \rightarrow \text{coker}(\gamma) \xrightarrow{\bar{\psi}'} \text{coker}(\tilde{\delta}) \xrightarrow{\bar{\phi}'} \text{coker}(\beta) \rightarrow 0.$$

It is clear by definition of \mathcal{E}^d that γ is an isomorphism. Thus,

$$0 \rightarrow 0 \rightarrow \ker(\tilde{\delta}) \xrightarrow{\bar{\phi}} \ker(\beta) \rightarrow 0 \rightarrow \text{coker}(\tilde{\delta}) \xrightarrow{\bar{\phi}'} \text{coker}(\beta) \rightarrow 0$$

is exact so $\bar{\phi}$ and $\bar{\phi}'$ are isomorphisms. Compare Proposition 5.4. \square

Corollary 5.6. *If \mathcal{F} is a GKM-sheaf, then $H^0(\mathcal{F})$ is reflexive.*

Proof. We see from (14) we have an exact sequence $0 \rightarrow H^0(\mathcal{F}) \rightarrow F_0 \rightarrow F_1$ where F_0 and F_1 are finitely generated free modules so $H^0(\mathcal{F})$ is a 2-syzygy, hence reflexive. \square

5.2 Local Cohomology of a GKM-Sheaf

Let \mathcal{F} be an GKM-sheaf over $Top(\Gamma) = \mathcal{V} \cup \mathcal{E}$. The set of vertices \mathcal{V} is an open set and the set of edges \mathcal{E} is a closed set, so we obtain a long exact sequence by Proposition 4.2

$$0 \rightarrow H^0(Top(\Gamma), \mathcal{F}) \rightarrow H^0(\mathcal{V}, \mathcal{F}) \rightarrow H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) \rightarrow H^1(Top(\Gamma), \mathcal{F}) \rightarrow \dots$$

Since \mathcal{V} is discrete, $H^i(\mathcal{V}, \mathcal{F}) = 0$ for all $i \geq 1$ and

$$0 \rightarrow H^0(Top(\Gamma), \mathcal{F}) \rightarrow H^0(\mathcal{V}, \mathcal{F}) \rightarrow H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) \rightarrow H^1(Top(\Gamma), \mathcal{F}) \rightarrow 0 \quad (15)$$

is exact.

Lemma 5.7. *The Godement chain complex for $H_{\mathcal{E}}^*(Top(\Gamma), \mathcal{F})$ is given by*

$$0 \rightarrow \prod_{e \in \mathcal{E}} \mathcal{F}_e \xrightarrow{\prod_{e \in \mathcal{E}} res_e} \prod_{e \in \mathcal{E}} \prod_{v \in I(e)} \mathcal{F}_v \rightarrow 0. \quad (16)$$

Proof. Applying Definition 3 and Lemma 5.3 we have

$$\begin{aligned} \Gamma_{\mathcal{E}}(C^0\mathcal{F}) &= \{s \in \Gamma(C^0\mathcal{F}) \mid s_v = 0, \forall v \in \mathcal{V}\} = \prod_{e \in \mathcal{E}} \mathcal{F}_e \\ \Gamma_{\mathcal{E}}(C^1\mathcal{F}) &= \{s \in \Gamma(C^1\mathcal{F}) \mid s_v = 0, \forall v \in \mathcal{V}\} = \prod_{e \in \mathcal{E}} \prod_{v \in I(e)} \mathcal{F}_v. \end{aligned}$$

and the boundary map $\Gamma_{\mathcal{E}}(C^0\mathcal{F}) \rightarrow \Gamma_{\mathcal{E}}(C^1\mathcal{F})$ is the natural one. \square

Proposition 5.8. *If Γ is a GKM-hypergraph and \mathcal{F} is a GKM-sheaf on $Top(\Gamma)$, then*

$$H^0(\mathcal{V}, \mathcal{F}) \cong \bigoplus_{v \in \mathcal{V}} \mathcal{F}_v, \text{ and}$$

$$H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) \cong \bigoplus_{e \in \mathcal{E}^{nd}} \text{coker}(res_e).$$

Proof. Since \mathcal{V} is discrete, $H^0(\mathcal{V}) = \prod_{v \in \mathcal{V}} \mathcal{F}(\{v\}) = \bigoplus_{v \in \mathcal{V}} \mathcal{F}_v$. By Lemma 5.7 we get

$$H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) = \text{coker}\left(\prod_{e \in \mathcal{E}} res_e\right) = \prod_{e \in \mathcal{E}^{nd}} \text{coker}(res_e).$$

By Definition 2, $\text{coker}(res_e) \neq 0$ only for e in a finite subset $\mathcal{E}^{nd} \subseteq \mathcal{E}$ so

$$\prod_{e \in \mathcal{E}^{nd}} \text{coker}(res_e) \cong \bigoplus_{e \in \mathcal{E}^{nd}} \text{coker}(res_e).$$

\square

6 Geometric meaning of GKM-sheaf cohomology

Theorem 6.1. *Let X be a finite T -CW complex. Then*

$$H_T^*(X) \cong H^0(\mathcal{F}_X)$$

if and only if $H_T^(X)$ is reflexive.*

Proof. Suppose $H_T^*(X) \cong H^0(\mathcal{F}_X)$. By Corollary 5.6, we conclude that $H_T^*(X)$ is reflexive.

Conversely, suppose that $H_T^*(X)$ is reflexive. By Theorem 2.2, the Chang-Skjelbred sequence

$$0 \rightarrow H_T^*(X) \rightarrow H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0)$$

is exact, so $H_T^*(X) \cong \ker(\delta)$ and $\ker(\delta) \cong H^0(\mathcal{F}_X)$ by Proposition 3.1. \square

Lemma 6.2. *Let $X'_1 \subseteq X_1$ be the union of path components that do not intersect X_0 . Suppose $H_T^*(X)$ is torsion free. Then $X'_1 = \emptyset$.*

Proof. Observe that X_1 can be written as follows:

$$X_1 = \bigcup_{H \leq T} X^H$$

where the union is indexed by codimension one subtori H . Since $H_T^*(X)$ is torsion free, $H_T^*(X^H)$ is free by Lemma 3.3. By the Localization theorem 2.1, every path component of X^H must intersect X_0 so $X^H \cap X'_1 = \emptyset$. We conclude $X'_1 = \emptyset$. \square

Lemma 6.3. *Suppose $H_T^*(X)$ is torsion free. Then $H^0(\mathcal{V}, \mathcal{F}_X) \cong H_T^*(X_0)$ and $H_{\mathcal{E}}^1(\text{Top}(\Gamma), \mathcal{F}_X) \cong H_T^{*+1}(X_1, X_0)$.*

Proof. The vertices $v \in \mathcal{V}$ corresponds path components of X_0 , so

$$H^0(\mathcal{V}, \mathcal{F}) = \bigoplus_{v \in \mathcal{V}} \mathcal{F}_v = \bigoplus_{v \in \mathcal{V}} H_T^*(v) = H_T^*(X_0).$$

The hyperedges $e \in \mathcal{E}$ for which $a(e) = \alpha$ correspond to path components of $X^{\ker(\alpha)}$ that intersect non-trivially with X_0 . Combine Lemma 3.4 with Lemma 6.2 to get

$$H_T^*(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H_T^*(e, e^T). \quad (17)$$

Claim 6.1. *If $H_T^*(X)$ is torsion free, then $\mathcal{F}_X(U_e) \cong H_T^*(e)$.*

Proof. Recall that by definition $\mathcal{F}_X(U_e) := H_T^*(e)/t$ where t is the torsion submodule so it is enough to show $H_T^*(e)$ is torsion free for all $e \in \mathcal{E}$. Since $H_T^*(X)$ is a submodule of finitely generated free R -module, it is torsion free. Apply Lemma 3.3. \square

The restriction morphism $res_e : \mathcal{F}(U_e) \rightarrow \mathcal{F}(I(e))$ is identical with the natural map $H_T^*(e) \rightarrow H_T^*(e^T)$ which is injective by the Localization Theorem 2.1. The long exact sequence for the pair (e, e^T) implies

$$H_T^{*+1}(e, e^T) = \text{coker}(res_e).$$

Applying Proposition 5.8, we have

$$H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) = \bigoplus_{e \in \mathcal{E}^{nd}} H_T^{*+1}(e, e^T).$$

Combining with (17), we conclude

$$H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}) \cong H_T^{*+1}(X_1, X_0).$$

□

Proof of Theorem 1.1. Since $H_T^*(X)$ is reflexive, Theorem 6.1 implies that

$$0 \rightarrow H^0(\mathcal{F}_X) \rightarrow H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0)$$

is exact. From Lemma 6.3 and (15) we have an isomorphism of exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^*(X) & \longrightarrow & H_T^*(X_0) & \xrightarrow{\delta} & H_T^{*+1}(X_1, X_0) & \longrightarrow & \text{coker}(\delta) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & H^0(\mathcal{F}_X) & \longrightarrow & H^0(\mathcal{V}, \mathcal{F}_X) & \longrightarrow & H_{\mathcal{E}}^1(Top(\Gamma), \mathcal{F}_X) & \longrightarrow & H^1(\mathcal{F}_X) & \longrightarrow & 0 \end{array}$$

□

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